Malaysian Journal of Mathematical Sciences 12(1): 63–84 (2018)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Commuting graphs and their generalized complements

Bhat, K. A and Sudhakara, G.

Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, India

E-mail:sudhakaraadiga@yahoo.co.in
* *Corresponding author*

Received: 4 April 2017
Accepted: 15 January 2018

ABSTRACT

In this paper we consider a graph G, a partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ and the generalized complements G_k^P and $G_{k(i)}^P$ with respect to the partition P. We derive conditions to be satisfied by P so that G commutes with its generalized complements. Apart from the general characterization, we also obtain conditions on $P = \{V_1, V_2, \ldots, V_k\}$ so that G commutes with its generalized complements for certain classes of graphs namely complete graphs, cycles and generalized wheels. In the process we obtain a commuting decomposition of regular complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ in terms of a Hamiltonian cycle and its k-complement. We also get a commuting decomposition of a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ in terms of a generalized wheel and its k-complement, where $n_1, n_2, \ldots, n_k$ satisfy some conditions.

Keywords: Adjacency matrix, Graphical, Matrix product, k- complement, k(i)- complement, Commutativity, Graph decomposition.

1. Introduction

Graphs considered in this paper are simple, undirected, and without self-loops. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. For any two vertices v_i and $v_j, i \neq j$ in the graph G, $v_i \sim_G v_j$ denotes that the vertices are adjacent in the graph G, and $v_i \nsim_G v_j$ denotes the vertices are not adjacent in the graph G. The suffix G in the notations $\sim_G$ and $\nsim_G$ are conveniently ignored if the graph under discussion is clearly understood. The notation $A(G)$ denotes the adjacency matrix of the graph G.

In an attempt to generalize the concept of complement of a graph G, Sampathkumar and Pushpalatha (1998) and Sampathkumar et al. (1998) have introduced the concept of G_k^P and $G_{k(i)}^P$ with respect to a partition P of $V(G)$ (Formal definitions of G_k^P and $G_{k(i)}^P$ will be given later). Several results appeared in literature about these complements, for example Sampathkumar and Pushpalatha (1998), Sampathkumar et al. (1998), Sampathkumar and Pushpalatha (1996), Sudhakara (2002), Sumathi and Brinda (2015).

Akbari and Herman (2007) and Akbari et al. (2009) have obtained the results on decomposition of complete graphs K_n and complete bipartite graphs $K_{n,n}$ into commuting perfect matchings and commuting Hamiltonian cycles. In Akbari et al. (2009), graphical matrix is defined as symmetric $(0,1)$- matrix with diagonal entries equal to zero. In the same paper centralizer of a graph G was defined and authors have obtained results on the number of elements in the centralizer of a cycle and complete graph on n vertices. In the paper by Manjunatha Prasad et al. (2013), graphicality of product of adjacency matrices $A(G)$ and $A(H)$ of graphs G and H was dealt and in Manjunatha Prasad et al. (2014), graphicality of product of adjacency matrices, where the product is taken over Z_2 was discussed. In Arathi Bhat et al. (2016), graphicality of the product $A(G)B(G)$ was derived, where $B(G)$ is the $(0,1)$ incidence matrix of the graph G.

In this paper, we derive properties of partition P of $V(G)$ of size k $(< n)$ such that $A(G)$ commutes with $A(G_k^P)$ and $A(G_{k(i)}^P)$ and we also obtain some of the particular elements in the centralizer of G which are derived graphs of G. And we obtain a commuting decomposition of regular complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ in terms of a Hamiltonian cycle and its k-complement. We also get a commuting decomposition of a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ in terms of a generalized wheel and its k-complement, where $n_1, n_2, \ldots, n_k$ satisfy some conditions.

Definitions of the G_k^P and the $k(i)$-complement $G_{k(i)}^P$ with respect to a partition P of $V(G)$ of size k, are given below.

Definition 1.1. *Sampathkumar and Pushpalatha (1998) Let G be a graph and $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$. The k-complement G_k^P, $(k \geq 2)$ of G with respect to P is defined as follows: For all V_i and V_j in $P, i \neq j$, remove the edges between V_i and V_j, and add the edges which are not in G.*

Definition 1.2. *Sampathkumar et al. (1998) Let G be a graph and $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$ of order $k \geq 1$. For each set V_r in P, remove the edges of G inside V_r and add the edges of $\overline{G}$, (the complement of G) joining the vertices of V_r. The graph $G_{k(i)}^P$ thus obtained is called the $k(i)$-complement of G with respect to the partition P.*

Definition 1.3. *Manjunatha Prasad et al. (2013) (GH path) Given graphs G and H on the same set of vertices $\{v_1, v_2, \ldots, v_n\}$, two vertices v_i and v_j $(i \neq j)$ are said to have a GH path from v_i to v_j, if there exists a vertex v_k, different from v_i and v_j, such that $v_i \sim_G v_k$ and $v_k \sim_H v_j$.*

Remark 1.4. *Let G and H be two graphs on the same set of vertices, say $\{v_1, v_2, \ldots, v_n\}$. Then G and H commute with each other if and only if for every two vertices v_i and $v_j, i \neq j, 1 \leq i, j \leq n$, the number of GH paths from v_i to v_j is same as number of HG paths from v_i to v_j.*

Readers are referred to West (1996) for all the elementary notations and definitions not described but used in this paper.

2. Commutativity of a graph G and its k-complement G_k^P

In this section, we characterize the graph G and the partition P of $V(G)$ such that the graph G commutes with its k-complement G_k^P.

Let G be a graph and $P = \{V_1, V_2, \ldots, V_k\}$ be the partition of $V(G)$. Consider a vertex $v \in V_i, i = 1, 2, \ldots, k$. Then the i-degree of v, defined with respect to the partition P of $V(G)$ is the degree of v in the graph induced by V_i, i.e., $\langle V_i \rangle$. And o-degree of v with respect to P is the number of vertices in $V(G) \setminus V_i$ which are adjacent to v in G.

Since every square matrix commutes with the zero matrix of the same size, the case that the graph H is totally disconnected and $A(H)$ is a zero matrix, is considered as trivial. In the further discussion in this paper we consider only

the graphs, adjacency matrix of which is non zero.
The following theorem characterizes the graph G, the partition $P = \{V_1, V_2, \dots, V_k\}$ of $V(G)$ for which graphs G and G_k^P with respect to the partition P, commute with each other.

Theorem 2.1. *Let $P = \{V_1, V_2, \dots, V_k\}$ be a partition of the vertex set $V(G)$ and let G_k^P be the k-complement of G with respect to the partition P. The graphs G and G_k^P commute with each other if and only if the partition P satisfies the following properties.*

(i) For every i, $1 \leq i \leq k$ and for every two vertices u and v in V_i, o-degree of u is same as o-degree of v.

(ii) For every two vertices $u \in V_i$ and $v \in V_j$, $1 \leq i, j \leq k$ and $i \neq j$,
$|A| + |B| = |C| + |D|$, where
$A = \{x \mid x \in V_i \text{ or } x \in V(G) \setminus (V_i \cup V_j) \text{ with } x \sim_G u \text{ and } x \nsim_G v\}$,
$B = \{x \in V_j \mid x \sim_G u \text{ and } x \sim_G v\}$,
$C = \{y \mid y \in V_j \text{ or } x \in V(G) \setminus (V_i \cup V_j) \text{ with } y \nsim_G u \text{ and } y \sim_G v\}$,
$D = \{y \in V_i \mid y \sim_G u \text{ and } y \sim_G v\}$
and $|X|$ represents the cardinality of set X.

Proof. Let the graph G and its k-complement G_k^P commute with each other. By the Remark 1.4, for every two vertices u and v of G, the number of GG_k^P paths from u to v is same as the number of $G_k^P G$ paths from u to v.

To prove that the partition satisfies conditions (i) and (ii) we consider the following two cases.

Case (i): Vertices u and v are in the same partite set say V_i, $1 \leq i \leq k$. By the definition of G_k^P, for any vertex w in V_i which is adjacent to both u and v, there is both GG_k^P and $G_k^P G$ paths from u to v through w. In all other possible cases, there is neither GG_k^P path nor $G_k^P G$ path from u to v. So, we consider vertex w which is outside V_i. If this vertex w is such that $u \sim_G w \nsim_G v$, then there is a GG_k^P path from u to v and hence there exists a vertex x outside V_i such that $u \nsim_G x \sim_G v$ which counts for $G_k^P G$ path from u to v.

Since G and G_k^P commute with each other, o-degree of u is same as o-degree of v. This proves (i).

Case (ii): Suppose that $u \in V_i$ and $v \in V_j$, $1 \leq i, j \leq k$, $i \neq j$. Then following are the ways of getting GG_k^P paths from u to v:

(a) corresponding to every vertex w in V_j which is adjacent to both u and v in G,
(b) corresponding to every vertex w either in V_i or in $V(G) \setminus (V_i \cup V_j)$ which is adjacent to u and non adjacent to v.

Similarly, we get $G_k^P G$ paths from u to v in the following ways:
(a') corresponding to every vertex w in V_i which is adjacent to both u and v in G,
(b') corresponding to every vertex w either in V_j or in $V(G) \setminus (V_i \cup V_j)$ which is adjacent to v and non adjacent to u.

Since G and G_k^P commute with each other, by the Remark 1.4, and by the above discussion, (ii) follows. Conversely, when the condition (i) and (ii) are satisfied, the number of GG_k^P paths is same as number of $G_k^P G$ paths between every two vertices u and v in G. Hence by Remark 1.4, the graphs G and G_k^P commute with each other. □

In the following, we give an example to demonstrate the above theorem. We consider a graph G, a partition P of $V(G)$ of size 2, satisfying conditions (i) and (ii) of Theorem 2.1. It can be verified that $A(G)A(G_2^P) = A(G_2^P)A(G)$ by computing both the products.

Example 2.2.

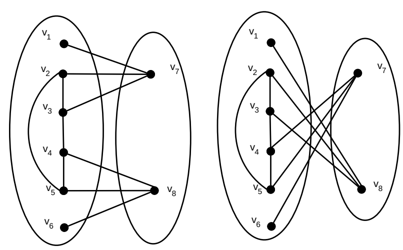

Figure 1: Graphs G and G_2^P satisfying $A(G)A(G_2^P) = A(G_2^P)A(G)$.

In the following section we investigate the existence of a partition P of $V(G)$ such that G commutes with G_k^P when G is taken from certain classes of graphs. We consider the class of trees, complete graphs, cycles and generalized wheels.

2.1 Trees

In this section we prove that for a tree G there exists no partition P of $V(G)$ such that G commutes with G_k^P, for any $k,\ 2 \leq k < n$.

Theorem 2.3. *Let G be a tree with n vertices. Then, for any positive integer $k \geq 2$, there exists no partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ of size k, such that G commutes with G_k^P.*

Proof. If possible, let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$ of size $k \geq 2$ such that G commutes with G_k^P.

Let u be a pendant vertex and v be the vertex adjacent to u in G. Since G is not K_2, (for $G = K_2$, $A(G_2^P)$ is a zero matrix), v is a vertex of degree at least 2.

If u and v are in the same partite set, say $V_i, 1 \leq i \leq k$, then, by (i) of Theorem 2.1, every vertex adjacent to v must be in V_i only. Likewise, every vertex adjacent to any vertex in V_i must lie in V_i. Effectively, all the vertices of G are in V_i, which is not possible, since $k \geq 2$. Therefore, u and v are in two different partite sets say, $u \in V_1$ and $v \in V_2$.

Consider a vertex w adjacent to v. If either $w \in V_2$ or w belongs to a partite set other than V_1 and V_2, say V_3, then there is at least one GG_k^P path but there is no $G_k^P G$ path from v to u. Hence, by Remark 1.4, it follows that all the vertices which are adjacent to v are in V_1.

Let w be any vertex in $V_1, w \neq u$ which is adjacent to v. Now, for any $x, x \neq v$ which is adjacent to w, either when $x \in V_1, x \in V_2$ or x is outside $V_1 \cup V_2$, one can show that there are different numbers of GG_k^P and $G_k^P G$ paths between two suitably chosen vertices, which is not possible. Therefore, there is no x adjacent to any vertex w in $N(v)$. Which implies $k = 2$ and the only possible tree is the star $K_{1,n-1}$. In which case G_2^P is a zero matrix, and the case is trivial. This completes the proof. □

2.2 Complete Graphs

Here we show the existence of a partition P of $V(K_n)$ such that K_n commutes with $(K_n)_k^P$ if and only if n is not a prime number.

Theorem 2.4. *Let G be the complete graph on n vertices. Then there exists a positive integer $k \geq 2$, and a partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$, such that*

G commutes with G_k^P, if and only if n is not a prime number.

Proof. Consider the complete graph $G = K_n$ and a partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ of order k, with $|V_i| = n_i,\ 1 \le i \le k$.

Then $A(G)$ can be viewed as,

$$A(G) = \begin{pmatrix} (J-I)_{n_1\times n_1} & J_{n_1\times n_2} & J_{n_1\times n_3} & \cdots & J_{n_1\times n_k} \\ J_{n_2\times n_1} & (J-I)_{n_2\times n_2} & \cdots & \cdots & J_{n_2\times n_k} \\ J_{n_3\times n_1} & J_{n_3\times n_2} & \cdots & \cdots & J_{n_3\times n_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{n_k\times n_1} & J_{n_k\times n_2} & \cdots & \ddots & (J-I)_{n_k\times n_k} \end{pmatrix}$$

with respect to the above $A(G)$, $A(G_k^P)$ becomes,

$$A(G_k^P) = \begin{pmatrix} (J-I)_{n_1\times n_1} & 0_{n_1\times n_2} & 0_{n_1\times n_3} & \cdots & 0_{n_1\times n_k} \\ 0_{n_2\times n_1} & (J-I)_{n_2\times n_2} & \cdots & \cdots & 0_{n_2\times n_k} \\ 0_{n_3\times n_1} & 0_{n_3\times n_2} & \cdots & \cdots & 0_{n_3\times n_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n_k\times n_1} & 0_{n_k\times n_2} & \cdots & \ddots & (J-I)_{n_k\times n_k} \end{pmatrix}.$$

Then the product $A(G)A(G_k^P)$ and the product $A(G_k^P)A(G)$ are given as follows:

$$A(G)A(G_k^P) = \begin{pmatrix} (J-I)^2_{n_1\times n_1} & J_{n_1\times n_2}(J-I)_{n_2\times n_2} & \cdots & J_{n_1\times n_k}(J-I)_{n_k\times n_k} \\ J_{n_2\times n_1}(J-I)_{n_1\times n_1} & (J-I)^2_{n_2\times n_2} & \cdots & J_{n_2\times n_k}(J-I)_{n_k\times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ J_{n_k\times n_1}(J-I)_{n_1\times n_1} & J_{n_k\times n_2}(J-I)_{n_2\times n_2} & \ddots & (J-I)^2_{n_k\times n_k} \end{pmatrix},$$

$$A(G_k^P)A(G) = \begin{pmatrix} (J-I)^2_{n_1\times n_1} & (J-I)_{n_1\times n_1}J_{n_1\times n_2} & \cdots & (J-I)_{n_1\times n_1}J_{n_1\times n_k} \\ (J-I)_{n_2\times n_2}(J_{n_2\times n_1}) & (J-I)^2_{n_2\times n_2} & \cdots & (J-I)_{n_2\times n_2}J_{n_2\times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ (J-I)_{n_k\times n_k}J_{n_k\times n_1} & (J-I)_{n_k\times n_k}J_{n_k\times n_2} & \ddots & (J-I)^2_{n_k\times n_k} \end{pmatrix}.$$

If the graphs G and G_k^P commute, then $J_{n_1\times n_2}(J-I)_{n_2\times n_2} = (J-I)_{n_1\times n_1}J_{n_1\times n_2}$. Which implies $(n_2-1)J_{n_1\times n_2} = (n_1-1)J_{n_1\times n_2}$, or $n_2 = n_1$. Proceeding like this $n_i = n_j$ for every $i = 1, 2, \ldots, k$. Therefore n is a multiple of k and n has to be a composite number.

Conversely, when n is a composite number, taking $\frac{n}{k}$ vertices in each partite set i.e., by considering $|V_i| = |V_j|$, for every i and j, $1 \leq i, j \leq k$, and taking the k-complement with respect to the above partition, we can retrace the steps above to show that G commutes with G_k^P. □

Remark 2.5. *In paper Manjunatha Prasad et al. (2014), while discussing about the graphical nature of the modulo 2 product $A(G)A(H)$ of the adjacency matrices of graphs G and H, authors have observed that the commutativity of $A(G)$ and $A(H)$ is required for the symmetry of the product matrix $A(G)A(H)$. The other essential property is that for every $i = 1, 2, \ldots, n$, there are even number of vertices v_k such that $v_i \sim Gv_k$ and $v_k \sim Hv_i$ which guarantees the zero diagonal.*

Now, suppose that K_n and $(K_n)_k^P$ commute with each other. If each $|V_i|, 1 \leq i \leq k$ is an odd integer i.e., if i-degree of each vertex is even, then we observe that when the multiplication is taken over Z_2, $A(G)A((G)_k^P)$ is always graphical.

2.3 Cycles

In this section we show that, a partition P of $V(C_n)$ with the property that C_n commutes with $(C_n)_k^P$ exists if and only if n is not a prime number.

Let C_n be a cycle on n vertices $v_1, v_2, \ldots, v_n$. Let $v_i \sim_G v_{i+1},\ i = 1, 2, \ldots, n-1,\ v_n \sim_G v_1$. Consider the $k-$complement $(C_n)_k^P$ of C_n with respect to some partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(C_n)$ of size $k \geq 2$. From (i) of Theorem 2.1, two graphs G and G_k^P commute if for every $i = 1, 2, \ldots, k$, all the vertices in V_i have the same o-degree.

Therefore, for every $i = 1, 2, \ldots, k$, V_i is either union of K_2's or totally disconnected. In the following theorem we prove that if C_n commutes with $(C_n)_k^P$, then $\langle V_i \rangle$ is totally disconnected.

Theorem 2.6. *Let $G = C_n$ be a cycle on n vertices $v_1, v_2, \ldots, v_n$ with $v_i \sim_G v_{i+1},\ i = 1, 2, \ldots, n-1,\ v_n \sim_G v_1$. Let G_k^P be of G with respect to some partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ of size $k \geq 2$. If G commutes with G_k^P then induced subgraphs $\langle V_i \rangle,\ i = 1, 2, \ldots, k$ are totally disconnected.*

Proof. Let G commute with G_k^P with respect to some partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ of size $k \geq 2$. Then we show that there is no partite set say $V_i, 1 \leq i \leq k$, such that $\langle V_i \rangle$ is either a K_2 or union of K_2's.

Suppose the edge $(v_1, v_2) \in V_1$ and let G_k^P, with respect to a partition $P = \{V_1, V_2, \ldots, V_k\}$ $(k \geq 2)$ of $V(G)$ commute with G.

Let both the vertices $v_3, v_n \in V_2$. Then from the vertex v_n to v_2 there is at least one GG_k^P path. But from v_2 to v_n there exists no GG_k^P path. Therefore v_3 and v_n cannot be in the same partite set.

Let $v_3 \in V_2$ and $v_n \in V_3$. Then from v_1 to v_n there is a GG_k^P path through v_2 and in order to get a GG_k^P path from v_n to v_1, the vertex v_{n-1} must be either in V_2 or in V_3 or lies outside $V_1 \cup V_2 \cup V_3$, say V_4.

In all of the above three cases, there are two GG_k^P paths and one $G_k^P G$ path from v_n to v_2, which by Remark1.4, is not possible.

Hence, when G commutes with G_k^P, $\langle V_i \rangle$ is totally disconnected for every $i = 1, 2, \ldots, k$. □

The following theorem gives all possible values of n and k and the partition P of $V(G)$, such that $G = C_n$ commutes with G_k^P.

Theorem 2.7. *Let $G = C_n$ be a cycle on n vertices $v_1, v_2, \ldots, v_n$ with $v_i \sim_G v_{i+1}$, $i = 1, 2, \ldots, n-1$, $v_n \sim_G v_1$, and let G_k^P be k-complement of G with respect to some partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ of size $k \geq 2$. G commutes with G_k^P if and only if each $\langle V_i \rangle$ is totally disconnected and $v_{lk+i} \in V_i$, $1 \leq i \leq k$ and $0 \leq l \leq \frac{n}{k} - 1$, n is a multiple of k.*

Proof. From the Theorem 2.6, if G commutes with G_k^P, then $\langle V_i \rangle$ is totally disconnected for every $i = 1, 2, \ldots, k$. Let $v_1 \in V_1$, $v_2 \in V_2$.

Suppose v_3 belong to V_1, then v_n belong to either V_2 or $V(G) \setminus (V_1 \cup V_2)$, say $v_n \in V_3$. The vertex $v_n \in V_2$ for, if it is in V_3, then from v_1 to v_2 there is a GG_k^P path but there is no GG_k^P path from v_2 to v_1.

Now the vertex $v_{n-1} \in V_1$ or V_3. But if $v_{n-1} \in V_3$, then from v_2 to v_{n-1} there are two GG_k^P paths, but from v_{n-1} to v_2 there is at most one GG_k^P path. Therefore $v_{n-1} \in V_1$. Proceeding like this, all vertices with odd index belong to V_1 and remaining vertices belong to V_2. Therefore when $v_n \in V_2$, $k = 2$ and n is a multiple of 2.

Suppose $v_3 \in V_3$ and $v_n \in V_3$, then the vertex v_4 can belong to V_1 or V_2 or outside $V_1 \cup V_2 \cup V_3$, say V_4 .

If $v_4 \in V_4$ or V_2, then from v_3 to v_1 there is a GG_k^P path through v_4, but from v_1 to v_3 there is no GG_k^P path. Therefore $v_4 \in V_1$.

Similarly we can prove that $v_{n-1}, v_5 \in V_2$ and so on. Therefore, $v_n, v_3 \in V_3 \implies k = 3$ and n is a multiple of 3. The vertices $v_{lk+i} \in V_i, 1 \leq i \leq 3,\ 0 \leq l \leq \frac{n}{3} - 1$.

Now suppose the vertex $v_i \in V_i, 1 \leq i \leq r$, and $v_n \in V_r$, then we prove that the vertex $v_{r+j} \in V_j, 1 \leq j \leq r$.

Suppose $v_{r+j} \notin V_j, 1 \leq j \leq r$, then one can show that there are different numbers of GG_k^P and $G_k^P G$ paths between two suitably chosen vertices, which is not possible.

Continuing like this, we get that the vertex $v_{lk+i} \in V_i,\ 1 \leq i \leq k$ and $0 \leq l \leq \frac{n}{k} - 1$ and n is a multiple of k.

Conversely, let $\langle V_i \rangle$ be totally disconnected with $v_{lk+i} \in V_i,\ 1 \leq i \leq k$ and $0 \leq l \leq \frac{n}{k} - 1$. Then in G_k^P with respect to the partition $P = \{V_1, V_2, \ldots, V_k\}$, a vertex v_i is non adjacent to $v_{i-1}, v_{i+1}, v_{k+i}, v_{2k+i}, \ldots, v_{lk+i}$ and adjacent to all the remaining vertices. Since there are $\frac{n}{k}$ vertices in every partite set G_k^P is regular with regularity $n - 2 - \frac{n}{k}$.

To show that $A(G)$ and $A(G_k^P)$ commute with each other, we show that both of them are circulant. Since G is a cycle $A(G)$ is circulant.

Consider i^{th} row of $A(G_k^P)$. The zero entries in this row are at the positions $(i, i-1), (i, i+1), (i, k+i), (i, 2k+i), \ldots, (i, lk+i)$. For all these pairs (i, j), $j - i + 1$ are given by, $0, 2, k+1, 2k+1, \ldots, lk+1$. The first row of $A(G_k^P)$ has zero entries at exactly the above column positions. Hence we get, $(A(G_k^P))_{i,j} = (A(G_k^P))_{1,j-i+1}$ for every i and j. Therefore, by definition, $A(G_k^P)$ is circulant. Since every two circulant matrices commute with each other, G commutes with G_k^P. □

Remark 2.8. *In paper Manjunatha Prasad et al. (2013), while discussing about the graphical nature of the product $A(G)A(H)$ of the adjacency matrices of graphs G and H, authors have observed that the commutativity of $A(G)$ and $A(H)$ is required for the symmetry of the product matrix $A(G)A(H)$. The other two essential properties are as follows. The graph H should be a subgraph of $\overline{G}$ which guarantees the zero diagonal and between every two vertices v_i and v_j, there can be at most one GH path from v_i to v_j and when there is one GH path* *from v_i to v_j then there is exactly one GH path from v_j to v_i, which guarantees that every entry is either 0 or 1.*

Now, suppose that $A(G)$ and $A(G_k^P)$ commute with each other. Since G is a cycle and degree of any vertex is two in G, between any two vertices there can be at most two GG_k^P paths. Thus, any entry of $A(G)A(G_k^P)$ is 0,1 or 2. And also, since every set V_i, $1 \leq i \leq k$, is independent, the diagonal of $A(G)A(G_k^P)$ has all entries equal to zero. Hence if there is no entry which is two, then $A(G)A(G_k^P)$ is graphical.

Therefore if multiplication is taken over Z_2, then $A(G)A(G_k^P)$ is always graphical.

When $k = 2$ and $|V_i| \geq 4, 1 \leq i \leq k$, then at least one entry in $A(G)A(G_k^P)$ is 2. Similarly, if $k \geq 3$ and $|V_i| \geq 2$ $1 \leq i \leq k$, then at least one entry in $A(G)A(G_k^P)$ is 2. In all such cases, with respect to usual multiplication $A(G)A(G_k^P)$ is not graphical.

Now consider the remaining cases.

Case (i): When $k \geq 3$ and $|V_i| = 1, 1 \leq i \leq k$. In this case, k will be equal to n and G_k^P is same as $\overline{G}$ and the corresponding results are well settled in paper Manjunatha Prasad et al. (2013).

Case (ii): When k=2 and $|V_i| \leq 3, 1 \leq i \leq k$. There are 2 cases. One, when $k = 2, |V_1| = |V_3| = 3$, in which case the graph G is C_6 and G_2^P is a 1-regular graph. And $A(G)A(G_k^P)$ is graphical with realizing graph of product is the David graph.

The other case, when $k = 2$ and $|V_1| = |V_2| = 2$ corresponds to the cycle C_4 and the corresponding G_2^P is totally disconnected. Therefore $A(G_2^P)$ is the zero matrix of order four, which is a trivial case.

In the following, we give an example to demonstrate the above remark. We consider a graph $G = C_9$, G_3^P with respect to a partition P of $V(G)$ of size 3, satisfying the conditions given in Theorem 2.7, and the graph Γ, where $A(\Gamma) = A(G)A(G_3^P)(mod\ 2)$. It can be verified that $A(G)A(G_3^P)(mod\ 2) = A(\Gamma)$ by computing the product.

Example 2.9.

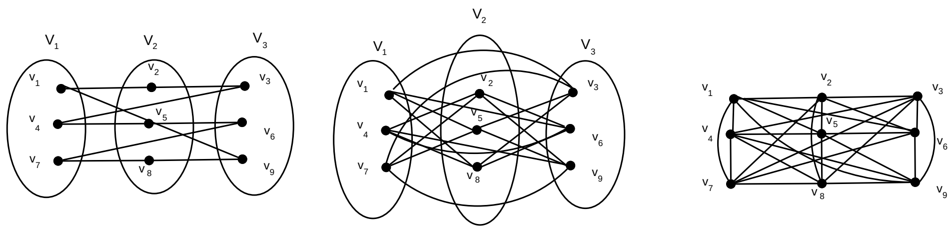

Figure 2: Graphs $G = C_9$, G_3^P and Γ with $A(\Gamma) = A(G)A(G_3^P)(mod\ 2)$.

Remark 2.10. *Let G, H and Γ be the graphs defined on the same set of vertices. According to Theorem 7 of Manjunatha Prasad et al. (2013), when $A(G)A(H) = A(\Gamma)$, degree of a vertex in the product graph Γ is given by $deg_\Gamma v_i = deg_G v_i . deg_H v_i$. Therefore when C_n commutes with $(C_n)_k^P$, row sum of the product $A(C_n)A((C_n)_k^P)$ is equal to $2(n - \frac{n}{k} - 2)$.*

2.4 Generalized Wheels

The Generalized Wheel $W_{m,n} = \overline{K_m} + C_n$ has m central vertices (vertices of $\overline{K_m}$) and n peripheral vertices (vertices of C_n). Every central vertex is adjacent to all the peripheral vertices.

In this section, we show the existence of a partition P of size k of $V(W_{m,n})$, with the property that $W_{m,n}$ commutes with $(W_{m,n})_k^P$.

Consider the generalized wheel $G = W_{m,n}$. Let $v_1, v_2, \ldots, v_n$ be the n vertices on the cycle with $v_i \sim_G v_{i+1}, 1 \leq i \leq n-1$, $v_n \sim_G v_1$ and let $v'_1, v'_2, \ldots, v'_m$ be the m central vertices. Partition the n vertices on the cycle into l partite sets $\{V_1, V_2, \ldots, V_l\}$ and m central vertices into $r = k - l$ partite sets $\{V_{l+1}, V_{l+2}, \ldots, V_{l+r} = V_k\}$. Let $|V_{l+i}| = m_i$ $l \leq i \leq r = k - l$ with $m_1 + m_2 + \ldots + m_r = m$.

By considering central vertices in the beginning and the vertices on the cycle afterwards, we can view $A(W_{m,n})$ and $A((W_{m,n})_k^P)$, as follows:

$$A(W_{m,n}) = \begin{pmatrix} O_{m\times m} & J_{m\times n} \\ J_{n\times m} & A(C_n)_{n\ \times n} \end{pmatrix},$$
$$A((W_{m,n})_k^P) = \begin{pmatrix} A(H_r^P)_{m\times m} & O_{m\times n} \\ O_{n\times m} & A((C_n)_l^P)_{n\ \times n} \end{pmatrix},$$

where $A(H_r^P)_{m\times m}$ is the adjacency matrix of r-complement of m central vertices with respect to the r partition and $A((C_n)_l^P)_{n\times n}$ is the adjacency matrix of l-complement of the n vertices on the cycle with respect to the l partition.

Theorem 2.11. *Let $v_1, v_2, \ldots, v_n$ be the n vertices on the cycle C_n and let $v_1', v_2', \ldots, v_m'$ be the m central vertices of $W_{m,n}$. Let the n vertices on the cycle be partitioned into l partite sets and m central vertices be partitioned into $r = k - l$ partite sets. The graphs $W_{m,n}$ and $(W_{m,n})_k^P$ with respect to the partition $P = \{V_1, V_2, \ldots, V_k\}$, commute if and only if the graph $W_{m,n}$ and the partition P satisfies the following properties;*

(i) each $\langle V_i\rangle$, $1 \le i \le k$ is totally disconnected with $v_{tl+i} \in V_i,\ 1 \le i \le l$ and $0 \le t \le \frac{n}{l} - 1$ and n is a multiple of l,

(ii) $m_i = \frac{1}{r-1}(n - \frac{n}{l} - 2)$, where $|V_{l+i}| = m_i$, $1 \le i \le r$.

Proof. The product $A(W_{m,n})A((W_{m,n})_k^P)$ and $A((W_{m,n})_k^P)A(W_{m,n})$ are given as follows;

$$A(W_{m,n})A((W_{m,n})_k^P) = \begin{pmatrix} O_{m\times m} & J_{m\times n}A((C_n)_l^P)_{n\times n} \\ J_{n\times m}A(H_r^P)_{m\times m} & A(C_n)_{n\ \times n}A((C_n)_l^P)_{n\times n} \end{pmatrix},$$
$$A((W_{m,n})_k^P)A(W_{m,n}) = \begin{pmatrix} O_{m\times m} & A(H_r^P)_{m\times m}J_{m\times n} \\ A((C_n)_l^P)_{n\times n}J_{n\times m} & A((C_n)_l^P)_{n\ \times n}A(C_n)_{n\times n} \end{pmatrix}.$$

From the above, we get, when $W_{m,n}$ commutes with $(W_{m,n})_k^P$, C_n must commute with $(C_n)_l^P$. Therefore, by Theorem 2.7, it follows that $\langle V_i\rangle$ is totally disconnected $1 \le i \le l$ and $v_{tl+i} \in V_i$, $0 \le t \le \frac{n}{l} - 1$ and n is a multiple of l. The graph $(C_n)_l^P$ is a regular with regularity $n - \frac{n}{l} - 2$.

Now $A(H_r^P) = \begin{pmatrix} O_{m_1\times m_1} & J_{m_1\times m_2} & \cdots & J_{m_1\times m_r} \\ J_{m_2\times m_1} & 0_{m_2\times m_2} & \cdots & J_{m_2\times m_r} \\ \vdots & \vdots & \ddots & \vdots \\ J_{m_r\times m_1} & J_{m_r\times m_2} & \cdots & 0_{m_r\times m_r} \end{pmatrix}$.

$$A(H_r^P)_{m\times m}J_{m\times n} = \begin{pmatrix} (m_2+m_3+\ldots+m_r)J_{m_1\times n} \\ (m_1+m_3+\ldots+m_r)J_{m_2\times n} \\ \vdots \\ (m_1+m_2+\ldots+m_{r-1})J_{m_r\times n} \end{pmatrix},$$

and $J_{m\times n}A((C_n)_l^P)_{n\times n} = (n - \frac{n}{l} - 2)J_{m\times n}$.
When $W_{m,n}$ commutes with $(W_{m,n})_k^P$,
$A(H_r^P)_{m\times m}J_{m\times n} = J_{m\times n}A((C_n)_l^P)_{n\times n}$.
Which implies, $m_i = m_j$ $1 \leq i \leq r$, and $m_i = \frac{1}{r-1}(n - \frac{n}{l} - 2)$.

Conversely, if the partition P of $V(W_{m,n})$ satisfies both the conditions of the theorem, then we can retrace the steps above to show that $W_{m,n}$ commutes with $(W_{m,n})_k^P$. □

We observe that for a given value of n, there exist many values of m and vice versa such that $W_{m,n}$ commutes with $(W_{m,n})_k^P$. We show the above fact in the following two examples. In the Example 2.12, we consider $n = 8$ and find all possible values of m and the corresponding graphs $W_{m,n}$. And in Example 2.13, we consider $m = 4$ and find all possible values of n and the corresponding graphs $W_{m,n}$.

Example 2.12. *For $n = 8$, as n is a multiple of l, l can take the values 2 or 4.*
Consider the case $l = 2$. Then $m_i = \frac{1}{r-1}(2)$ implies that r can take the values either 2 or 3. Therefore when $r = 2, m = 4$ results in $W_{4,8}$ with $k = 4$. And $r = 3, m = 3$ results in $W_{3,8}$ with $k = 5$.
Consider the case $l = 4$. Then $m_i = \frac{1}{r-1}(4)$ and hence r can take the values 2,3 or 5. When $r = 2, m = 8$ results in $W_{8,8}$ with $k = 6$. And when $r = 3, m = 6$ results in $W_{6,8}$ with $k = 7$. Finally, $r = 5, m = 5$ results in $W_{5,8}$ with $k = 9$.

Example 2.13. *When $m = 4$, r can take the values 2 or 4.*
Consider the case r=2. Then $m_i = 2$ and $l = \frac{n}{n-4}$. Therefore n can take the values either 5,6 or 8.
When $n = 5$, $l = 5$ results in $W_{4,5}$, with $k = 7$. When $n = 6$, $l = 3$ results in $W_{4,6}$ with $k = 5$. And when $n = 8, l = 2$ results in $W_{4,8}$, with $k = 4$.
Consider the case $r = 4$. Then $m_i = 1$ and $l = \frac{n}{n-5}$. Therefore n can take the values 6 or 10.
When $n = 6$, $l = 6$ results in $W_{4,6}$, with $k = 10$. And when $n = 10$, $l = 2$ results in $W_{4,10}$ with $k = 6$.

Remark 2.14. *Let $G = W_{m,n}$. Suppose that G and G^P_k commute with each other, then we observe that, by Remark 2.5, when the multiplication is taken over Z_2, $A(G)A(G^P_k)$ is always graphical.*

3. Commuting decomposition of Complete k-partite graphs

A decomposition of a graph G is a collection of subgraphs $H_1, H_2, \ldots, H_k$ that partitions the edges of G. That is, for all i and j, $\bigcup\limits_{1\leq i\leq k} H_i = G$ and $E(H_i) \cap E(H_j) = \Phi$ for $i \neq j$.

This section deals with commuting decomposition of a complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ into a subgraph and its k-complement. Theorem 3.1 explains the commuting decomposition of $K_{n_1,n_2,\ldots,n_k}$ into a cycle C_n and its k-complement $(C_n)^P_k$. Theorem 3.2 gives the commuting decomposition of $K_{n_1,n_2,\ldots,n_k}$ into a generalized wheel $W_{m,n}$ and its k-complement $(W_{m,n})^P_k$. In both the cases, we consider the partition P to be the k-partition of the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$.

Theorem 3.1. *Let G be a regular complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$, where $n_i = \dfrac{n}{k}$ for $i = 1, 2, \ldots, k$ with respect to a partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$. Then the graph G is decomposable into two commuting subgraphs of G, one of which is C_n and the other one is its k-complement with respect to the same partition $P = \{V_1, V_2, \ldots, V_k\}$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertices of G. Let the partition $P = \{V_1, V_2, \ldots, V_k\}$ of the regular k-partite graph be taken as follows. The vertices $v_{lk+i} \in V_i, 1 \leq i \leq k,\ 0 \leq l \leq \dfrac{n}{k} - 1$. Then observe that the graph $G = K_{n_1,n_2,\ldots,n_k}$ has a Hamiltonian cycle C_n on the vertices $v_1, v_2, \ldots, v_n$ taken in that order. Let this subgraph be denoted by H_1. Remove the edges of C_n from G. Let the resulting graph be H_2. Consider $(C_n)^P_k$ with respect to the same partition $P = \{V_1, V_2, \ldots, V_k\}$. Two vertices in $(C_n)^P_k$ are adjacent if and only if they are adjacent in H_2. Hence $(C_n)^P_k$ is same as H_2. Which implies that $K_{n_1,n_2,\ldots,n_k} = H_1 \cup H_2 = C_n \cup (C_n)^P_k$ with $E(H_1) \cap E(H_2) = \Phi$. Hence $H_1 = C_n$ and $H_2 = (C_n)^P_k$ form a decomposition of $K_{n_1,n_2,\ldots,n_k}$. From the Theorem 2.7, $(C_n)^P_k$ is a circulant graph. Because C_n is also circulant, C_n commutes with $(C_n)^P_k$. Therefore $C_n, (C_n)^P_k$ form a commuting decomposition of G. □

Theorem 3.2. *Let G be a complete k-partite graph $K_{n_1,n_2,\ldots,n_l,n_{l+1},\ldots,n_{l+r=k}}$ having the vertex set $\{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_m\}$ with respect to the partition $P = \{V_1, V_2, \ldots, V_l, V_{l+1}, \ldots, V_{l+r=k}\}$ of $V(G)$, where $n_i = \frac{n}{l}$ for $i = 1, 2, \ldots, l$ and $n_i = \frac{1}{r-1}(n - \frac{n}{l} - 2)$ for $i = l+1, l+2, \ldots, k$. Then the graph G is decomposable into two commuting subgraphs of G, one of which is $W_{m,n}$ and the other one is its k-complement with respect to the same partition $P = \{V_1, V_2, \ldots, V_k\}$.*

Proof. Let the vertices $v_1, v_2, \ldots, v_n$ be such that the vertices $v_{lr+i} \in V_i, 1 \leq i \leq l,\ 0 \leq r \leq \frac{n}{l} - 1$. And the vertices $v'_1, v'_2, \ldots, v'_m$ belong to the remaining partite sets $\{V_{l+1}, V_{l+2}, \ldots, V_k\}$. Define the subgraph H_1 of G as follows. H_1 is a spanning subgraph of G with $E(H_1)$ consisting of a cycle with vertices $v_1, v_2, \ldots, v_n$ in that order and all the edges joining each v'_i $1 \leq i \leq m$ to every vertex on the above cycle. Observe that the subgraph H_1 is a generalized wheel $W_{m,n}$.
Remove the edges of the subgraph H_1 from G. Let the resulting graph be H_2. Consider $(W_{m,n})^P_k$ with respect to the same partition $P = \{V_1, V_2, \ldots, V_k\}$. Two vertices in $(W_{m,n})^P_k$ are adjacent if and only if they are adjacent in H_2. Hence $(W_{m,n})^P_k$ is same as H_2. Which implies that $K_{n_1,n_2,\ldots,n_k} = H_1 \cup H_2 = W_{m,n} \cup (W_{m,n})^P_k$ with $E(H_1) \cap E(H_2) = \Phi$. Hence $H_1 = W_{m,n}$ and $H_2 = (W_{m,n})^P_k$ form a decomposition of $K_{n_1,n_2,\ldots,n_k}$. From the Theorem 2.11, $W_{m,n}$ commutes with $(W_{m,n})^P_k$. Therefore $W_{m,n}, (W_{m,n})^P_k$ form a commuting decomposition of G. □

In Akbari et al. (2009), authors have obtained all positive integral values of n for which the graph $K_{n,n}$ is decomposable into commuting Hamiltonian cycles. We observe that the commuting decomposition of $K_{n_1,n_2,\ldots,n_k}, n_i = \frac{n}{k}$ for $i = 1, 2, \ldots, k$ into a cycle C_n and its k-complement becomes a commuting decomposition of two Hamiltonian cycles only when $(C_n)^P_k \cong C_n$. When this is true, the corresponding vertices have same degree in C_n and $(C_n)^P_k$. Since C_n is regular with regularity 2 and $(C_n)^P_k$ is regular with regularity $(n - \frac{n}{k} - 2)$, we get $n - \frac{n}{k} - 2 = 2$. Which gives either $k = 2, n = 8$ and the graph is $K_{4,4}$ or $k = 5, n = 5$ and the graph is $K_{1,1,1,1,1}$ or $k = 3, n = 6$ and the graph is $K_{2,2,2}$. But when $k = 3, n = 6$, $(C_n)^P_k$ is union of two C'_3s. Therefore $K_{4,4}$ and $K_{1,1,1,1,1}$ are the only graphs that can be decomposed into two commuting Hamiltonian cycles in terms of C_n and its k-complement $(C_n)^P_k$.

4. Commutativity of a graph G and its $k(i)$-complement

In this section we derive the conditions to be satisfied by the partition P of $V(G)$ in order that the graphs G and $G^P_{k(i)}$ commute with each other. We state the result in the form of a theorem, the proof of which is similar to that of Theorem 2.1, and hence is omitted.

Theorem 4.1. *Let $P = \{V_1, V_2, \ldots, V_k\}(k \geq 2)$ be a partition of the vertex set $V(G)$ and let $G^P_{k(i)}$ be the $k(i)$-complement of G with respect to the partition P. The graphs G and $G^P_{k(i)}$ commute with each other, if and only if the partition P satisfies the following properties.*

(i) For every i, $1 \leq i \leq k$ and for every two vertices u and v in V_i, the i-degree of u is same as the i-degree of v.

(ii) For every two vertices $u \in V_i$ and $v \in V_j$, $1 \leq i, j \leq k$ and $i \neq j$,
$|A| + |B| = |C| + |D|$, where
$A = \{x \in V_i \mid x \sim_G u$ and $x \sim_G v\}$,
$B = \{x \in V_j \mid x \sim_G u$ and $x \nsim_G v\}$,
$C = \{y \in V_j \mid y \sim_G u$ and $y \sim_G v\}$,
$D = \{y \in V_i \mid y \nsim_G u$ and $y \sim_G v\}$
and $|X|$ represents the cardinality of set X.

In the following, we give an example to demonstrate the above theorem. We consider a graph G, a partition P of $V(G)$ of size 2, satisfying conditions (i) and (ii) of 4.1. It can be verified that $A(G)A(G^P_{2(i)}) = A(G^P_{2(i)})A(G)$ by computing both the products (Figure 3).

Example 4.2.

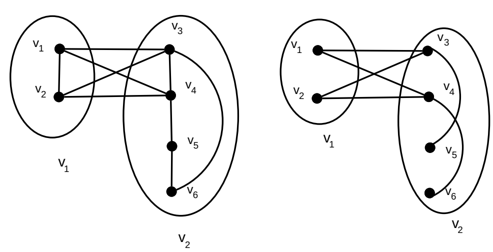

Figure 3: Graphs G and $G^P_{2(i)}$

In the following section we investigate the existence of a partition P of $V(G)$ such that G commutes with $G^P_{k(i)}$ when G is taken from certain classes of graphs. We consider the classes of complete graphs, cycles and generalized wheels.

4.1 Complete Graphs

Here we show the existence of a partition P of $V(K_n)$ such that K_n commutes with $(K_n)^P_{k(i)}$ if and only if n is not a prime number.

Theorem 4.3. *Let G be the complete graph on n vertices. Then there exists a positive integer $k \geq 2$, and a partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$, such that G commutes with $G^P_{k(i)}$, if and only if n is not a prime number.*

Proof. Consider the complete graph $G = K_n$ and a partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ with $|V_i| = n_i,\ 1 \leq i \leq k$.
Then $A(G)$ can be viewed as,

$$A(G) = \begin{pmatrix} (J-I)_{n_1\times n_1} & J_{n_1\times n_2} & J_{n_1\times n_3} & \cdots & J_{n_1\times n_k} \\ J_{n_2\times n_1} & (J-I)_{n_2\times n_2} & \cdots & \cdots & J_{n_2\times n_k} \\ J_{n_3\times n_1} & J_{n_3\times n_2} & \cdots & \cdots & J_{n_3\times n_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{n_k\times n_1} & J_{n_k\times n_2} & \cdots & \ddots & (J-I)_{n_k\times n_k} \end{pmatrix}.$$

With respect to the above $A(G)$, $A(G^P_{k(i)})$ becomes,

$$A(G^P_{k(i)}) = \begin{pmatrix} 0_{n_1\times n_1} & J_{n_1\times n_2} & J_{n_1\times n_3} & \cdots & J_{n_1\times n_k} \\ J_{n_2\times n_1} & 0_{n_2\times n_2} & \cdots & \cdots & J_{n_2\times n_k} \\ J_{n_3\times n_1} & J_{n_3\times n_2} & \cdots & \cdots & J_{n_3\times n_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{n_k\times n_1} & J_{n_k\times n_2} & \cdots & \ddots & 0_{n_k\times n_k} \end{pmatrix}.$$

Then the product $A(G)A(G^P_{k(i)})$ and the product $A(G^P_{k(i)})A(G)$ are given as follows;

$$A(G)A(G^P_{k(i)}) = \begin{pmatrix} J_{n_1\times n_2}J_{n_2\times n_1} + \cdots + J_{n_1\times n_k}J_{n_k\times n_1} & (J-I)_{n_1\times n_1}J_{n_1\times n_2} + J_{n_1\times n_3}J_{n_3\times n_2} + \cdots + J_{n_1\times n_k}J_{n_k\times n_2} \\ (J-I)_{n_2\times n_2}J_{n_2\times n_1} + \cdots + J_{n_2\times n_k}J_{n_k\times n_1} & J_{n_2\times n_1}J_{n_1\times n_2} + J_{n_2\times n_3}J_{n_3\times n_2} + \cdots + J_{n_2\times n_k}J_{n_k\times n_2} \\ \vdots & \vdots \end{pmatrix}$$

$$A(G^P_{k(i)})A(G) = \begin{pmatrix} J_{n_1\times n_2}J_{n_2\times n_1} + \cdots + J_{n_1\times n_k}J_{n_k\times n_1} & J_{n_1\times n_2}(J-I)_{n_2\times n_2} + J_{n_1\times n_3}J_{n_3\times n_2} + \cdots + J_{n_1\times n_k}J_{n_k\times n_2} \\ J_{n_2\times n_1}(J-I)_{n_1\times n_1} + \cdots + J_{n_2\times n_k}J_{n_k\times n_1} & J_{n_2\times n_1}J_{n_1\times n_2} + J_{n_2\times n_3}J_{n_3\times n_2} + \cdots + J_{n_2\times n_k}J_{n_k\times n_2} \\ \vdots & \vdots \end{pmatrix}$$

If the graphs G and $G^P_{k(i)}$ commute then, $J_{n_1\times n_2}(J-I)_{n_2\times n_2} = (J-I)_{n_1\times n_1}J_{n_1\times n_2}$.

Which implies $(n_2 - 1)J_{n_1 \times n_2} = (n_1 - 1)J_{n_1 \times n_2}$, or $n_2 = n_1$. Proceeding like this, $n_i = n_j$ for every i and j, $1 \le i, j \le k$. Therefore n is a multiple of k and n has to be a composite number.
Conversely, if n is a composite number, taking $\frac{n}{k}$ vertices in each partite set i.e., by considering $|V_i| = |V_j|$ for every i and j, $1 \le i, j \le k$, and taking the k-complement with respect to the above partition, we can retrace the steps above to show that G commutes with $G^P_{k(i)}$. □

Remark 4.4. *Let $G = K_n$. Suppose G and $G^P_{k(i)}$ commute with each other and if o-degree of all the vertices is an even integer, then by Remark 2.5, if the multiplication is taken over Z_2, then $A(G)A(G^P_{k(i)})$ is always graphical.*

4.2 Cycles

In this section we show that, a partition P of $V(C_n)$ with the property that C_n commutes with $(C_n)^P_{k(i)}$ exists if and only if n is not a prime number.
Let C_n be a cycle on n vertices $v_1, v_2, \ldots, v_n$. Let $v_i \sim_G v_{i+1},\ i = 1, 2, \ldots, n-1,\ v_n \sim_G v_1$.
Consider the $k(i)-$complement $(C_n)^P_{k(i)}$ of C_n with respect to some partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(C_n)$ of size $k \ge 2$. From (i) of Theorem 4.1, two graphs G and $G^P_{k(i)}$ commute if for every $i = 1, 2, \ldots, k$, all the vertices in V_i have the same i-degree.
Therefore when C_n commutes with $(C_n)^P_{k(i)}$, the graph induced by V_i is either union of K_2's or totally disconnected.
In the following theorem we prove that when C_n commutes with $(C_n)^P_{k(i)}$, then $\langle V_i \rangle$ is totally disconnected. The proof of this theorem is similar to that of Theorem 2.6 and hence is omitted.

Theorem 4.5. *Let $G = C_n$ be a cycle on n vertices $v_1, v_2, \ldots, v_n$ with $v_i \sim_G v_{i+1},\ i = 1, 2, \ldots, n-1,\ v_n \sim_G v_1$. Let $G^P_{k(i)}$ be $k(i)$-complement of G with respect to some partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ of size $k \ge 2$. If G commutes with $G^P_{k(i)}$ then induced subgraphs $\langle V_i \rangle,\ i = 1, 2, \ldots, k$ are totally disconnected.*

The following theorem gives all possible values of n and k and the partition P of $V(G)$, such that $G = C_n$ commutes with $G^P_{k(i)}$. The proof of this theorem is similar to that of Theorem 2.7, and hence is omitted.

Theorem 4.6. *Let $G = C_n$ be a cycle on n vertices $v_1, v_2, \ldots, v_n$ with $v_i \sim_G v_{i+1},\ i = 1, 2, \ldots, n-1,\ v_n \sim_G v_1$, and let $G^P_{k(i)}$ be $k(i)$-complement of G with respect to some partition $P = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ of size $k \ge 2$.*

G commutes with $G^P_{k(i)}$ if and only if each $\langle V_i \rangle$ is totally disconnected and $v_{lk+i} \in V_i,\ 1 \le i \le k$ and $0 \le l \le \frac{n}{k} - 1$, n is a multiple of k.

Remark 4.7. *Let $G = C_n$. Suppose G and $G^P_{k(i)}$ commute with each other, then we observe by Remark 2.5 that, when the multiplication is taken over Z_2, $A(G)A(G^P_{k(i)})$ is always graphical.*

4.3 Generalized Wheels

In this section, we show the existence of a partition P of $V(W_{m,n})$ of size k, with the property that $W_{m,n} = \overline{K_m} + C_n$ commutes with $(W_{m,n})^P_{k(i)}$.
Let $v_1, v_2, \ldots, v_n$ be the n vertices on the cycle with $v_i \sim_G v_{i+1}, 1 \le i \le n-1$, $v_n \sim_G v_1$ and let $v'_1, v'_2, \ldots, v'_m$ be the m central vertices. Partition the n vertices on the cycle into l partite sets $\{V_1, V_2, \ldots, V_l\}$ and m central vertices into $r = k - l$ partite sets $\{V_{l+1}, V_{l+2}, \ldots, V_{l+r} = V_k\}$. Let $|V_{l+i}| = m_i$ $1 \le i \le r$ with $m_1 + m_2 + \ldots + m_r = m$.
By considering central vertices in the beginning and the vertices on the cycle afterwards, we can view $A(W_{m,n})$ and $(A(W_{m,n}))^P_{k(i)}$, as follows;

$$A(W_{m,n}) = \begin{pmatrix} O_{m\times m} & J_{m\times n} \\ J_{n\times m} & A(C_n)_{n\ \times n} \end{pmatrix},\ A((W_{m,n})^P_{k(i)}) = \begin{pmatrix} A(H^P_{r(i)})_{m\times m} & J_{m\times n} \\ J_{n\times m} & A((C_n)^P_{l(i)})_{n\times n} \end{pmatrix},$$

where $A(H^P_{r(i)})_{m\times m}$ is the adjacency matrix of r(i)-complement of m central vertices with respect to the r partition and $(A(C_n)^P_{l(i)})_{n\times n}$ is the adjacency matrix of l(i)-complement of the vertices on n cycle with respect to the l partition.

Theorem 4.8. *Let $v_1, v_2, \ldots, v_n$ be the n vertices on the cycle C_n and let $v'_1, v'_2, \ldots, v'_m$ be the m central vertices of $W_{m,n}$. Let the cycle vertices be partitioned into l partite set and m central vertices be partitioned into r partite sets so that $l + r = k$. The graphs $W_{m,n}$ and $(W_{m,n})^P_{k(i)}$ commutes if and only if the graph $W_{m,n}$ and the partition satisfies the following properties;*

(i) each $\langle V_i \rangle$, $1 \le i \le k$ is totally disconnected with $v_{tl+i} \in V_i,\ 1 \le i \le l$ and $0 \le t \le \frac{n}{l} - 1$ and n is a multiple of l.

(ii) $m_i = \frac{n}{l}$, where $|V_{l+i}| = m_i$, $1 \le i \le r$.

Proof. The product $A(W_{m,n})A((W_{m,n})^P_{k(i)})$ and $A((W_{m,n})^P_{k(i)})A(W_{m,n})$ are given as follows;

$$A(W_{m,n})A((W_{m,n})^P_{k(i)}) = \begin{pmatrix} J_{m\times n}J_{n\times m} & J_{m\times n}A((C_n)^P_{l(i)})_{n\times n} \\ J_{n\times m}A(H^P_{r(i)})_{m\times m} + A(C_n)_{n\times n}J_{n\times m} & J_{n\times m}J_{m\times n} + A(C_n)_{n\ \times n}A((C_n)^P_{l(i)})_{n\times n} \end{pmatrix}$$

$$A((W_{m,n})^P_{k(i)})A(W_{m,n}) = \begin{pmatrix} J_{m\times n}J_{n\times m} & A(H^P_{r(i)})_{m\times n}J_{m\times n} + J_{m\times n}A(C_n)_{n\times n} \\ (A(C_n)^P_{l(i)})_{n\ \times n}J_{n\times m} & J_{n\times m}J_{m\times n} + A((C_n)^P_{l(i)})_{n\ \times n}A(C_n)_{n\ \times n} \end{pmatrix}$$

As $W_{m,n}$ commutes with $(W_{m,n})^P_{k(i)}$, C_n must commute with $(C_n)^P_{l(i)}$. Therefore from Theorem 4.6, $\langle V_i\rangle,\ 1 \le i \le l$ is totally disconnected with $v_{tl+i} \in V_i\ , 0 \le t \le \frac{n}{l} - 1$ and n is a multiple of l. Also $(C_n)^P_{l(i)}$ is regular graph with regularity $\frac{n}{l} + 1$.

$$\text{Now } A(H^P_{r(i)}) = \begin{pmatrix} (J-I)_{m_1\times m_1} & 0_{m_1\times m_2} & \cdots & 0_{m_1\times m_r} \\ 0_{m_2\times m_1} & (J-I)_{m_2\times m_2} & \cdots & 0_{m_2\times m_r} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{m_r\times m_1} & 0_{m_r\times m_2} & \cdots & (J-I)_{m_r\times m_r} \end{pmatrix}$$

$A(H^P_{r(i)})_{m\times m}J_{m\times n} + J_{m\times n}A(C_n)_{n\times n} =$

$$\begin{pmatrix} (m_1-1)J_{m_1\times n} \\ (m_2-1)J_{m_2\times n} \\ \vdots \\ (m_r-1)J_{m_r\times n} \end{pmatrix} + 2J_{m\times n} = \begin{pmatrix} (m_1+1)J_{m_1\times n} \\ (m_2+1)J_{m_2\times n} \\ \vdots \\ (m_r+1)J_{m_r\times n} \end{pmatrix},$$

and $J_{m\times n}A((C_n)^P_{l(i)})_{n\times n} = (\frac{n}{l} + 1)J_{m\times n}$.

As $W_{m,n}$ commutes with $(W_{m,n})^P_{k(i)}$,

$A(H^P_{r(i)})_{m\times n}J_{m\times n} + J_{m\times n}A(C_n)_{n\times n} = J_{m\times n}((A(C_n))^P_{l(i)})_{n\times n}$

Which implies $m_i = m_j = \frac{n}{l},\ 1 \le i \le r$.

Conversely, if $W_{m,n}$ satisfies both the conditions of the theorem with respect to the k-complement of the partition P, then we can retrace the steps above to show that $W_{m,n}$ commutes with $(W_{m,n})^P_{k(i)}$. □

As in Example 2.12 and 2.13, we observe that for a given value of n, there exists more than one value of m and vice versa such that $W_{m,n}$ commutes with $(W_{m,n})^P_{k(i)}$.

Remark 4.9. *Suppose $A(W_{m,n})$ and $A((W_{m,n})^P_{k(i)})$ commute with each other and n is even. Then we observe by Remark 2.5, that when the multiplication is taken over Z_2, $A(W_{m,n})A((W_{m,n})^P_{k(i)})$ is always graphical.*

References

Akbari, S. and Herman, A. (2007). Commuting decompositions of complete graphs. *Journal of Combinatorial Designs*, 15(2):133–142.

Akbari, S., Moazami, F., and Mohammadian, A. (2009). Commutativity of the adjacency matrices of graphs. *Discrete Mathematics*, 309(3):595 – 600.

Arathi Bhat, K., Manjunatha Prasad, K., and Sudhakara, G. (2016). Some matrix equations of graphs. *Advances and Applications in Discrete Mathematics*, 17(1):29.

Manjunatha Prasad, K., Sudhakara, G., Sujatha, H. S., and Soumya, K. V. (2014). Matrix product(modulo-2) of graphs. *Indian Journal of Pure and Applied Mathematics*, 45(6):851–860.

Manjunatha Prasad, K., Sudhakara, G., Sujatha, H. S., and Vinay, M. (2013). *Matrix Product of Graphs*, pages 41–55. Springer India, India.

Sampathkumar, E. and Pushpalatha, L. (1996). Reconstruction of a graph of order p from its $(p-1)-$ complements. *Indian Journal of pure and applied Mathematics*, 27(5):435–441.

Sampathkumar, E. and Pushpalatha, L. (1998). Complement of a graph: A generalization. *Graphs and Combinatorics*, 14(4):377–392.

Sampathkumar, E., Pushpalatha, L., Venkatachalam, and Pradeep Bhat, G. (1998). Generalized complements of a graph. *Indian J. pure appl. Math*, 29(6):625–639.

Sudhakara, G. (2002). *Wheels, Cages and Cubes*, pages 251–259. Birkhäuser Basel, Basel.

Sumathi, P. and Brinda, T. (2015). Generalized two complement of set domination. *International Journal of Scientific and Technology Research*, 4(12):262–369.

West, D. (1996). *Introduction to Graph Theory.* Prentice Hall, United States.